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1998 J. Phys. A: Math. Gen. 31 2929

(http://iopscience.iop.org/0305-4470/31/12/016)

J. Phys. A: Math. Gen. **31** (1998) 2929–2940. Printed in the UK PII: S0305-4470(98)87764-0

Quantum Lorentz and braided Poincaré groups

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Received 23 September 1997

Abstract. Quantum Lorentz groups H admitting quantum Minkowski space V are selected. The natural structure of a quantum space $G = V \times H$ is introduced, defining a quantum group structure on G only for triangular H ($q = 1$). We show that it defines a braided quantum group structure on G for $|q| = 1$.

Introduction

Any example of a quantum Poincaré group [1] is constructed using one of the quantum Lorentz groups introduced in [2]. However, only very special cases of the latter (triangular deformations) can be used for this purpose. Cases related to the celebrated q-deformation of Drinfeld and Jimbo are, unfortunately, excluded. This is in fact a general feature of inhomogeneous quantum groups [3, 4].

It turned out recently that this obstacle can be circumvented, if one allows the deformed inhomogeneous group to be a braided quantum group rather than an ordinary quantum group. It means that the comultiplication is a morphism into a non-trivial crossed-product algebra rather than the usual product. It turns out that on the level of generators, the only non-trivial cross-relations are those for the translation coordinates. These results have been derived in our previous paper [5] for the case when the homogeneous part is the standard q-deformed (with $|q| = 1$) orthogonal quantum group $SO(p, p)$, $SO(p, p+1)$ [6] or $SO(p, p+2)$ [7]. The author has recently learned of the paper by Drabant [8] where results of similar type (without the reality condition) were obtained (see also [9, 10]).

In the present paper we study the case when the homogeneous part H is the Lorentz group. This case requires separate study, because we have the possibility of taking into account the complete classification of quantum deformations [2]. Another reason for a separate treatment is that we want to consider the 'more fundamental' simply connected $SL(2, \mathbb{C})$ group instead of $SO(1, 3)$.

The paper is organized as follows. In section 1 we recall non-triangular, deformation-type cases of quantum Lorentz group H. In section 2 we select those cases which have the corresponding quantum Minkowski space V (this happens for $|q| = 1$ or $q^2 \in \mathbb{R}$). In section 3 we construct a natural crossed 'Cartesian product' G of V and H (as quantum spaces). In section 4 we investigate conditions under which the natural formula for the comultiplication on generators defines a morphism of algebras, the product algebra being understood with suitable crossed (or braided) structure.

The same program on the Poisson level has already been presented in [5].

We conclude in section 5 with explicit commutation relations for the Minkowski space. Several proofs are relegated to an appendix.

0305-4470/98/002929+12\$19.50

1. Quantum Lorentz groups

We recall that the *-algebra $\mathcal{A} = \mathrm{Poly}\,(H)$ of polynomials on quantum $H = SL(2,\mathbb{C})$ is generated by the matrix elements of

$$u = (u^A_B)_{A,B=1,2} = \begin{pmatrix} u^1_1 & u^1_2 \\ u^2_1 & u^2_2 \end{pmatrix}$$

subject to the relations

$$u_1 u_2 E = E \qquad E' u_1 u_2 = E' \qquad X u_1 \overline{u}_2 = \overline{u}_1 u_2 X \tag{1}$$

where E, E' and X are described in [2, theorem 2.2]. Here the subscripts 1 and 2 refer to the position of a given object in the tensor product of the underlying 'arithmetic' vector space (in this case $\mathbb{C}^2$, with the standard basis e_1, e_2). For instance, the first equality means that $u^A_C u^B_D E^{CD} = E^{AB}$ (summation convention). We omit the subscripts when the object has only one natural position in a given situation (like E for instance). The complex conjugate $\overline{u}$ of u is given by

$$\overline{u} = (\overline{u}^{\overline{A}}_{\overline{B}})_{A,B=1,2} = \begin{pmatrix} (u^1_1)^* & (u^1_2)^* \\ (u^2_1)^* & (u^2_2)^* \end{pmatrix} \qquad \text{i.e. } \overline{u}^{\overline{A}}_{\overline{B}} = (u^A_B)^*.$$

The 'barred' indices refer to the complex conjugate basis $e_{\overline{1}} := \overline{e_1}$, $e_{\overline{2}} := \overline{e_2}$ in $\overline{\mathbb{C}^2}$.

With the standard comultiplication defined on generators by $\Delta u = uu'$ (primed coordinates refer to the *second copy* of H; in a less compact notation, $\Delta u^A_B = u^A_C \otimes u^C_B \in \mathcal{A} \otimes \mathcal{A}$), the above *-algebra becomes a Hopf *-algebra. (For instance, Δ preserves the relations $\Delta u_1 \Delta u_2 E = u_1 u'_1 u_2 u'_2 E = u_1 u_2 u'_1 u'_2 E = u_1 u_2 E = E$.)

In what follows we focus on non-triangular deformations. This means that

$$E = e_1 \otimes e_2 - q e_2 \otimes e_1 \qquad E' = e^2 \otimes e^1 - q^{-1} e^1 \otimes e^2 \qquad q \in \mathbb{C} \setminus \{0, i, -i\} \tag{2}$$

(the standard q-deformation) and X is given by [2, equations (13) or (15)], i.e. we have one of the following two cases:

1. $X = t^{\frac{1}{2}}(e^1_1 \otimes e^{\overline{1}}_{\overline{1}} + e^2_2 \otimes e^{\overline{2}}_{\overline{2}}) + t^{-\frac{1}{2}}(e^1_2 \otimes e^{\overline{2}}_{\overline{1}} + e^2_1 \otimes e^{\overline{1}}_{\overline{2}})$ for $0 < t \in \mathbb{R}$
2. $X = q^{\frac{1}{2}}(e^1_1 \otimes e^{\overline{1}}_{\overline{1}} + e^2_2 \otimes e^{\overline{2}}_{\overline{2}}) + q^{-\frac{1}{2}}(e^1_2 \otimes e^{\overline{2}}_{\overline{1}} + e^2_1 \otimes e^{\overline{1}}_{\overline{2}}) \pm q^{\frac{1}{2}} e^2_1 \otimes e^{\overline{2}}_{\overline{1}}$ for $0 < q \in \mathbb{R}$

(with the obvious notation for the matrix units $e^{\overline{1}}_{\overline{1}} := e_{\overline{1}} \otimes e^1$, etc).

Any matrix which intertwines $u_1 u_2$ with itself and satisfies the braid equation is proportional to

$$M := q\mathcal{P}' - q^{-1}\mathcal{P} \qquad \text{or} \qquad M^{-1} = q^{-1}\mathcal{P}' - q\mathcal{P} \tag{3}$$

where $\mathcal{P} := -(q+q^{-1})^{-1} E E'$ (the deformed antisymmetrizer) is the projection on E parallel to $\ker E'$ and $\mathcal{P}' := I - \mathcal{P}$ (the deformed symmetrizer). Conjugating $M^{\pm 1} u_1 u_2 = u_1 u_2 M^{\pm 1}$ we obtain $K^{\pm 1} \overline{u}_1 \overline{u}_2 = \overline{u}_1 \overline{u}_2 K^{\pm 1}$, where

$$K := \tau \overline{M} \tau = \overline{q} \mathcal{Q}' - \overline{q}^{-1} \mathcal{Q} \qquad \mathcal{Q} := \tau \overline{\mathcal{P}} \tau \quad \mathcal{Q}' := I - \mathcal{Q}. \tag{4}$$

Throughout the paper τ denotes the permutation in the tensor product.

2. Quantum Minkowski spaces

In order to discuss quantum Minkowski spaces that are covariant under the quantum Lorentz group, we consider the four-dimensional representation of the latter:

$$h := u_1\overline{u}_2 \qquad \text{i.e.}\ \ h^{A\overline{B}}_{C\overline{D}} := u^A_C \overline{u}^{\overline{B}}_{\overline{D}}. \tag{5}$$

Note that $\tau\overline{h}\tau = h$. This means that in a basis of elements self-adjoint with respect to the natural conjugation $x \mapsto \tau\overline{x}$ in $\mathbb{C}^2 \otimes \overline{\mathbb{C}^2}$, such as the basis of Pauli matrices $\sigma_j^{A\overline{B}}$ ($j = 0, \ldots, 3$), the matrix h has self-adjoint elements. In considerations which refer only to the four-dimensional ('vector') representation, it is often convenient to use exactly the components of h in the basis of Pauli matrices. These components will be denoted by h^j_k ($j, k = 0, \ldots, 3$). In the leg-numbering notation (as in equation (1)) we shall use bold subscripts for the four-dimensional case. For instance, the tensor square of h will be denoted either by $h_{12}h_{34}$ (referring to the spinor representation) or by $h_{\mathbf{1}}h_{\mathbf{2}}$ (referring to the vector representation).

Now we look for appropriate quadratic commutation relations defining the quantum Minkowski space. Here we use the standard method of dealing with 'quantum vector spaces'. The algebra of polynomials on quantum Minkowski space should be generated by four generators $x = (x^{A\overline{B}})_{A,B=1,2} = (x^j)_{j=0,\ldots,3}$ satisfying the reality condition

$$\tau\overline{x} = x \qquad (\text{i.e.}\ \ (x^{A\overline{B}})^* = x^{B\overline{A}} \quad \text{or} \quad (x^j)^* = x^j) \tag{6}$$

and some quadratic relations $Ax_{\mathbf{1}}x_{\mathbf{2}} = 0$, such that $\Delta_V x := hx'$ satisfies the same relations (note, that $\Delta_V x$ satisfies the reality automatically). The last requirement will be satisfied if A is an intertwiner of $h_{\mathbf{1}}h_{\mathbf{2}}$:

$$A\Delta_V x_{\mathbf{1}}\Delta_V x_{\mathbf{2}} = Ah_{\mathbf{1}}x_{\mathbf{1}}h_{\mathbf{2}}x_{\mathbf{2}} = Ah_{\mathbf{1}}h_{\mathbf{2}}x_{\mathbf{1}}x_{\mathbf{2}} = h_{\mathbf{1}}h_{\mathbf{2}}Ax_{\mathbf{1}}x_{\mathbf{2}} = 0 \tag{7}$$

(this is the key point of the method of [6]). It remains for us to choose an appropriate intertwiner: it should be a deformation of the antisymmetrizer (see also remark 2.2 below).

From $M^{\pm 1}$, $K^{\pm 1}$, X and X^{-1} we can build easily four intertwiners of $h_{\mathbf{1}}h_{\mathbf{2}} = h_{12}h_{34}$, namely

$$\hat{R}_{\pm} := X_{23}(M_{12}K^{\pm 1}_{34})X^{-1}_{23} \tag{8}$$

and their inverses. Each of them becomes the permutation in the classical limit.

Proposition 2.1. Matrices $\hat{R}_{\pm}$ satisfy the braid equation (with $\mathbb{C}^2 \otimes \overline{\mathbb{C}^2}$ being the elementary space).

The proof is given in appendix A.1.

Substituting equations (3), (4) into (8) we obtain the spectral decomposition

$$\hat{R}_{\pm} = X_{23}(q\overline{q}^{\pm 1}\mathcal{P}' \otimes \mathcal{Q}' + q^{-1}\overline{q}^{\mp 1}\mathcal{P} \otimes \mathcal{Q} - q\overline{q}^{\mp 1}\mathcal{P}' \otimes \mathcal{Q} - q^{-1}\overline{q}^{\pm 1}\mathcal{P} \otimes \mathcal{Q}')X^{-1}_{23}. \tag{9}$$

Since the projections $\mathcal{P}' \otimes \mathcal{Q}$, $\mathcal{P} \otimes \mathcal{Q}'$ are three-dimensional,

$$P^{(-)} := X_{23}(\mathcal{P}' \otimes \mathcal{Q} + \mathcal{P} \otimes \mathcal{Q}')X^{-1}_{23} \tag{10}$$

is a good candidate for the deformed antisymmetrizer. It is in fact easy to see that it becomes the classical antisymmetrizer in the classical limit.

Remark 2.2. It is not necessary to use the argument of a 'deformed antisymmetrizer'. In fact, there is a more straightforward (logical) approach. Note that the subspace V^* spanned by x^j is invariant with respect to H, and we are looking just for a six-dimensional invariant subspace of $V^* \otimes V^*$. It must be therefore the direct sum of the two three-dimensional irreducible subrepresentations in $V^* \otimes V^*$.

Definition 2.3. The *-algebra generated by $(x^j)^* = x^j$ (i.e. $(x^{A\overline{B}})^* = x^{B\overline{A}}$) and the relations

$$P^{(-)}x_{\mathbf{1}}x_{\mathbf{2}} = 0 \qquad (\text{i.e.} \quad P^{(-)}x_{12}x_{34} = 0) \tag{11}$$

is said to be the **-algebra of polynomials on quantum Minkowski space* (and denoted by Poly (V)) if it has the classical size (i.e. if the Poincaré–Birkhoff–Witt theorem holds).

Proposition 2.4. Quantum Minkowski spaces exist only for

$$|q| = 1 \qquad \text{or} \qquad \overline{q}^2 = q^2. \tag{12}$$

For the proof, see appendix A.2. Note that this result was conjectured in [11].

In what follows we assume one of two possibilities: $q = \overline{q}$ or $|q| = 1$ (we discard $q = -\overline{q}$ as not of deformation type).

Note that for $q = \overline{q}$

$$\hat{R}_+ = X_{23}(q^2\mathcal{P}' \otimes \mathcal{Q}' + q^{-2}\mathcal{P} \otimes \mathcal{Q} - \mathcal{P}' \otimes \mathcal{Q} - \mathcal{P} \otimes \mathcal{Q}')X_{23}^{-1} \tag{13}$$

$$\hat{R}_- = X_{23}(\mathcal{P}' \otimes \mathcal{Q}' + \mathcal{P} \otimes \mathcal{Q} - q^2\mathcal{P}' \otimes \mathcal{Q} - q^{-2}\mathcal{P} \otimes \mathcal{Q}')X_{23}^{-1} \tag{14}$$

and for $|q| = 1$

$$\hat{R}_+ = X_{23}(\mathcal{P}' \otimes \mathcal{Q}' + \mathcal{P} \otimes \mathcal{Q} - q^2\mathcal{P}' \otimes \mathcal{Q} - q^{-2}\mathcal{P} \otimes \mathcal{Q}')X_{23}^{-1} \tag{15}$$

$$\hat{R}_- = X_{23}(q^2\mathcal{P}' \otimes \mathcal{Q}' + q^{-2}\mathcal{P} \otimes \mathcal{Q} - \mathcal{P}' \otimes \mathcal{Q} - \mathcal{P} \otimes \mathcal{Q}')X_{23}^{-1} \tag{16}$$

hence for $q = \overline{q}$

$$\text{relations (11)} \quad \Longleftrightarrow \quad \hat{R}_- x_{\mathbf{1}}x_{\mathbf{2}} = x_{\mathbf{1}}x_{\mathbf{2}}$$

and for $|q| = 1$

$$\text{relations (11)} \quad \Longleftrightarrow \quad \hat{R}_+ x_{\mathbf{1}}x_{\mathbf{2}} = x_{\mathbf{1}}x_{\mathbf{2}}$$

which 'explains' why for $q = \overline{q}$ or $|q| = 1$ we obtain the appropriate size of the algebra generated by x, namely, different ways of ordering the polynomials of the third degree give the same result, due to the Yang–Baxter property of $\hat{R}_\pm$ (proposition 2.1):

$$R_{\mathbf{12}}R_{\mathbf{13}}R_{\mathbf{23}}x_{\mathbf{1}}x_{\mathbf{2}}x_{\mathbf{3}} = x_{\mathbf{3}}x_{\mathbf{2}}x_{\mathbf{1}} = R_{\mathbf{23}}R_{\mathbf{13}}R_{\mathbf{12}}x_{\mathbf{1}}x_{\mathbf{2}}x_{\mathbf{3}} \tag{17}$$

where $R = \tau\hat{R}_-$ for $\overline{q} = q$ and $R = \tau\hat{R}_+$ for $|q| = 1$.

3. The crossed product of Minkowski with Lorentz

In this section we shall introduce a crossed tensor product of Poly (H) and Poly (V) in such a way that the standard comultiplication

$$\Delta u = uu' \qquad \Delta x = x + hx' \tag{18}$$

preserves 'as much as possible' of the algebraic structure (preserves as many relations as possible). Technically (see theorem 3.1 below for the precise statement), we consider the universal *-algebra $\mathcal{B}$ generated by u^A_B and $x^j = (x^j)^*$, satisfying (1), (11) and the *cross relations*

$$x_{12}u_3 = Tu_1x_{23} \qquad (\text{i.e.} \quad x^{A\overline{B}}u^C_D = T^{A\overline{B}C}_{EK\overline{L}}u^E_D x^{K\overline{L}}) \tag{19}$$

for an appropriate matrix T, which we select after some discussion.

Note that the 'preservation of relations' by Δ means that Δu and Δx do satisfy the same relations as u and x. Let us check when it happens. Of course, Δu satisfies (1) as before. Since

$$\Delta x_{12}\Delta u_3 = (x_{12} + h_{12}x'_{12})u_3u'_3 = Tu_1x_{23}u'_1 + h_{12}u_3Tu'_1x'_{23}$$

(here we use $x'_{12}u_3 = u_3x'_{12}$) and

$$T\Delta u_1 \Delta x_{12} = Tu_1u'_1(x_{23} + h_{23}x'_{23}) = Tu_1x_{23}u'_1 + Tu_1h_{23}u'_1x'_{23},$$

Δx and Δu satisfy (19) if

$$Tu_1h_{23} = h_{12}u_3T \tag{20}$$

i.e. $T \in \mathrm{Mor}\,(u_1u_2\overline{u}_3, u_1\overline{u}_2u_3)$ (T intertwines $u_1u_2\overline{u}_3$ with $u_1\overline{u}_2u_3$). It means that

$$T = X_{23}S_{12} \tag{21}$$

for some $S \in \mathrm{Mor}\,(u_1u_2, u_1u_2)$, which we assume to be invertible.

The discussion of when Δ preserves (11) will be postponed to section 4.

By taking the star operation of (19), we obtain

$$\overline{u}_1x_{23} = X_{12}(\tau\overline{S}\tau)_{23}x_{12}\overline{u}_3 \tag{22}$$

(we have used the property $\tau\overline{X}\tau = X$); hence

$$x_{12}\overline{u}_3 = (\tau\overline{S}^{-1}\tau)_{23}X_{12}^{-1}\overline{u}_1x_{23}. \tag{23}$$

It follows that

$$x_{12}h_{34} = X_{23}S_{12}(\tau\overline{S}^{-1}\tau)_{34}X_{23}^{-1}h_{12}x_{34} \tag{24}$$

and the matrix governing the commutation of x and h has a similar structure to $\hat{R}_-$ in (8). This suggests that S should be proportional to M or M^{-1}. We shall show that this is indeed the case, if we require $\mathcal{B}$ to have a correct size.

Recall that a *crossed tensor product* of two algebras, $\mathcal{C}$ and $\mathcal{D}$, is the tensor product of the vector spaces $\mathcal{C} \otimes \mathcal{D}$ equipped with the multiplication

$$m = (m_{\mathcal{C}} \otimes m_{\mathcal{D}})(\mathrm{id} \otimes s \otimes \mathrm{id}) \tag{25}$$

$m_{\mathcal{C}}$ and $m_{\mathcal{D}}$ being the multiplication maps in $\mathcal{C}$ and $\mathcal{D}$, where $s: \mathcal{D} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{D}$ is a linear map satisfying

$$\begin{aligned}(\mathrm{id} \otimes s)(m_{\mathcal{D}} \otimes \mathrm{id}) &= (\mathrm{id} \otimes m_{\mathcal{D}})(s \otimes \mathrm{id})(\mathrm{id} \otimes s)\\ (s \otimes \mathrm{id})(\mathrm{id} \otimes m_{\mathcal{C}}) &= (m_{\mathcal{C}} \otimes \mathrm{id})(\mathrm{id} \otimes s)(s \otimes \mathrm{id})\end{aligned} \tag{26}$$

(this condition is equivalent to the associativity of m). For unital algebras we require additionally $s(I \otimes c) = c \otimes I$, $s(d \otimes I) = (I \otimes d)$ and for *-algebras, we require that $*_{12}*_{12} = \mathrm{id}$, where

$$*_{12} = s(* \otimes *)\tau. \tag{27}$$

Under these conditions $\mathcal{C} \otimes \mathcal{D}$ becomes a unital *-algebra (called the *crossed tensor product of* $\mathcal{C}$ and $\mathcal{D}$) and the inclusions $c \mapsto c \otimes I$, $d \mapsto I \otimes d$ are unital *-homomorphisms (see, for instance, [12]).

Theorem 3.1. If there exists a crossed tensor product of $*$-algebras $\mathrm{Poly}\,(H)$ and $\mathrm{Poly}\,(V)$, compatible with (19), i.e. such that

$$s(x^{A\overline{B}} \otimes u^C_D) = T^{A\overline{B}C}_{EK\overline{L}}u^E_D \otimes x^{K\overline{L}} \tag{28}$$

then it is unique. It exists if and only if

$$S = q^{-\frac{1}{2}}M \qquad \text{or} \qquad S = q^{\frac{1}{2}}M^{-1} \tag{29}$$

(the square roots are defined up to sign).

The proof is given in appendix A.3.

From now on we shall consider the case when $S = q^{-\frac{1}{2}}M$ (the second case in (29) is completely analogous). We can write equation (24) as

$$x_1 h_2 = \hat{W} h_1 x_2 \tag{30}$$

where

$$\hat{W} = \hat{R}_- \quad \text{for} \quad \overline{q} = q \qquad \hat{W} = q^{-1}\hat{R}_- \quad \text{for} \quad |q| = 1. \tag{31}$$

4. The Poincaré group with braided translations—only for $|q| = 1$

Now we can return to the problem of when Δ preserves (11), i.e. when $P^{(-)}x_1x_2 = 0$ implies $P^{(-)}\Delta x_1 \Delta x_2 = 0$. Assuming equation (11) holds, first two terms in

$$\Delta x_1 \Delta x_2 = (x_1 + h_1 x_1')(x_2 + h_2 x_2') = x_1 x_2 + h_1 x_1' h_2 x_2' + x_1 h_2 x_2' + h_1 x_1' x_2 \tag{32}$$

are obviously annihilated by $P^{(-)}$ (second, because $P^{(-)}h_1h_2x_1'x_2' = h_1h_2P^{(-)}x_1'x_2' = 0$). In the last term we shall need to commute x_1' with x_2. Normally they just commute, but it will be convenient to consider here the following more general situation:

$$x_1'x_2 = \hat{B}x_1x_2' \qquad \text{or} \qquad x_2'x_1 = Bx_1x_2' \qquad (\hat{B} = \tau B) \tag{33}$$

(for some matrix B). In particular, if $B = \mathrm{id}$, x_1' and x_2 commute. Note that this more general assumption does not affect previous results on the preservation of (1) and (19).

The sum of the last two terms in (32) is equal to

$$(\hat{W}h_1x_2x_1' + h_1x_1'x_2)^{jk} = \hat{W}^{jk}_{ab}h^a_c x^b x'^c + h^j_l B^{kl}_{bc} x^b x'^c = (\hat{W}^{jk}_{ab}\delta^l_c + \delta^j_a B^{kl}_{bc})h^a_l x^b x'^c$$

hence, finally, Δ preserves (11) when

$$P^{(-)}_{12}(\hat{W}_{12} + B_{23}) = 0. \tag{34}$$

Now, if $B = I$, then using equations (31) we see that the above equality is possible only for $q^2 = 1$.

This is one more manifestation of the fact that the standard q-deformation is not compatible with inhomogeneous groups.

On the other hand, if we could manage that

$$P^{(-)}(\hat{W} + \sigma I) = 0 \qquad \text{for some } \sigma \tag{35}$$

then $B = \sigma I$ satisfies (34). In this case Δ preserves (11) provided we consider the 'braiding'

$$x'^j x^k = \sigma x^k x'^j. \tag{36}$$

Taking into account that $P^{(-)}$ is a projection and a function of $\hat{W}$, condition (35) means that $P^{(-)}$ is a spectral projection of $\hat{W}$ corresponding to a single eigenvalue (equal to $-\sigma$). From equations (31), (14) and (16) it is clear that this is possible only for $|q| = 1$ and in this case $\sigma = q^{-1}$.

It is easy to check that equations (36) consistently define a crossed tensor product of $\mathcal{B}$ with itself. A triple crossed product is also naturally defined (one uses (36) between each pair) and the coassociativity holds. Concluding, we have a family of braided Poincaré groups, labelled by two parameters: $|q| = 1$ and $t > 0$.

The braided Poincaré group acts on the corresponding quantum Minkowski space in the braided sense: one has to take into account the non-trivial cross-relations (36) in the product.

5. Minkowski space

Here we present the defining relations (11) for the quantum Minkowski space corresponding to $|q| = 1$ and $t > 0$ explicitly:

$$\alpha\beta = tq\beta\alpha$$
$$\alpha\gamma = t^{-1}q\gamma\alpha$$
$$\beta\delta = tq\delta\beta$$
$$\gamma\delta = t^{-1}q\delta\gamma$$
$$\beta\gamma = \gamma\beta$$
$$[\alpha, \delta] = t^{-1}(q - q^{-1})\beta\gamma$$

and

$$\alpha^* = \alpha \qquad \delta^* = \delta \qquad \beta^* = \gamma \tag{37}$$

(cf equations (A14)–(A17) and (A22)). We have denoted the elements $x^{A\overline{B}}$ as follows:

$$x = \begin{pmatrix} x^{1\overline{1}} & x^{1\overline{2}} \\ x^{2\overline{1}} & x^{2\overline{2}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{38}$$

We may introduce a complex parameter $z := q/t \neq 0$. The corresponding quantum Minkowski space $\mathcal{M}_z$ is described by the $*$-algebra Poly $(\mathcal{M}_z)$ generated by the elements α, δ, γ satisfying

$$\alpha^* = \alpha \qquad \delta^* = \delta \qquad \gamma^*\gamma = \gamma\gamma^* \tag{39}$$

such that

$$\alpha\gamma = z\gamma\alpha$$
$$\gamma\delta = z\delta\gamma$$
$$[\alpha, \delta] = (z - \overline{z})\gamma^*\gamma.$$

The (Lorentz-invariant) *Minkowski length*, obtained as $E'_{12}(\tau\overline{E'})_{34}X^{-1}_{23}x_{12}x_{34}$, is a central element of Poly (V). In terms of the basic generators it equals

$$\frac{\alpha\delta}{2z} + \frac{\delta\alpha}{2\overline{z}} - \gamma^*\gamma.$$

Acknowledgments

The author is very much indebted to Professor S L Woronowicz for drawing attention to the problem and valuable discussions. The author acknowledges financial support under Polish KBN grant No 2 P301 020 07.

Appendix

A.1. Braid intertwiners for Minkowski

When we represent equations (8) on a diagram composed of elementary crossings X, X^{-1}, M and $K^{\pm 1}$, then it becomes clear that it is sufficient to prove the 'elementary moves'

$$X_{12}^{-1} M_{23} X_{12} = X_{23} M_{12} X_{23}^{-1} \tag{A1}$$

$$X_{12}^{-1} X_{23}^{-1} K_{12}^{\pm 1} = K_{23}^{\pm 1} X_{12}^{-1} X_{23}^{-1} \tag{A2}$$

$$K_{12}^{\pm 1} X_{23} X_{12} = X_{23} X_{12} K_{23}^{\pm 1} \tag{A3}$$

$$M_{12} X_{23}^{-1} X_{12}^{-1} = X_{23}^{-1} X_{12}^{-1} M_{23} \tag{A4}$$

$$X_{12} X_{23} M_{12} = M_{23} X_{12} X_{23} \tag{A5}$$

$$X_{12} K_{23}^{\pm} X_{12}^{-1} = X_{23}^{-1} K_{12}^{\pm 1} X_{23}. \tag{A6}$$

The equalities (A1), (A4) and (A5) are mutually equivalent. Also, the equalities (A2), (A3) and (A6) are mutually equivalent. Note that (A3) with a 'plus sign' is obtained by taking the complex conjugate of (A5). The 'minus sign' case is obtained by the complex conjugation of $X_{12} X_{23} M_{12}^{-1} = M_{23}^{-1} X_{12} X_{23}$, which is of course a simple consequence of (A5) since M^{-1}, is a polynomial of M.

Thus, it is sufficient to prove (A5). This equality is almost evident from [2] (it expresses the fact that X provides a representation of the standard q-commutation relations). Let us prove this in detail. Since M is a linear combination of I and EE', it is sufficient to show that

$$X_{12} X_{23} E_{12} E'_{12} = E_{23} E'_{23} X_{12} X_{23}.$$

From [2, equation (5)] we know that $X_{12} X_{23} E_{12} = c E_{23}$, and, analogously, $E'_{23} X_{12} X_{23} = d E'_{12}$ for some non-zero factors $c, d \in \mathbb{C}$. But $d = c$, since

$$(d E'_{12}) E_{12} = (E'_{23} X_{12} X_{23}) E_{12} = E'_{23} (X_{12} X_{23}) E_{12}) = E'_{23} (c E_{23})$$

and this completes the proof.

A.2. Selection of parameters for Minkowski space

We shall write relations (11) in explicit form. The two cases of X may be written in one formula:

$$X = e_1^1 \otimes e_1^{\overline{1}} + e_2^2 \otimes e_2^{\overline{2}} + t^{-1} (e_2^1 \otimes e_1^{\overline{2}} + e_1^2 \otimes e_2^{\overline{1}}) + \varepsilon e_1^2 \otimes e_1^{\overline{2}}$$

where $\varepsilon = 0$ in case 1 and $\varepsilon = \pm 1$, $t = q$ in case 2 (we have rescaled X for convenience). We have

$$X^{-1} = e_1^{\overline{1}} \otimes e_1^1 + e_2^{\overline{2}} \otimes e_2^2 + t (e_1^{\overline{2}} \otimes e_2^1 + e_2^{\overline{1}} \otimes e_1^2) - \varepsilon e_1^{\overline{2}} \otimes e_1^2.$$

Of course, equation (11) is equivalent to

$$(\mathcal{P}' \otimes \mathcal{Q}) X_{23}^{-1} x_{12} x_{34} = 0 \qquad \text{and} \qquad (\mathcal{P} \otimes \mathcal{Q}') X_{23}^{-1} x_{12} x_{34} = 0.$$

Using

$$\ker \mathcal{P}' = \langle E \rangle = \ker \langle E \rangle^\circ = \ker \langle e^{11}, e^{22}, e^{21} + q e^{12} \rangle$$

$$\ker \mathcal{Q} = \ker \tau \overline{E'} = \ker (\overline{q} e^{\overline{12}} - e^{\overline{21}}) \qquad \ker \mathcal{P} = \ker E' = \ker (q e^{21} - e^{12})$$

$$\ker \mathcal{Q}' = \langle \tau \overline{E} \rangle = \ker \langle \tau \overline{E} \rangle^\circ = \ker \langle e^{\overline{11}}, e^{\overline{22}}, \overline{q} e^{\overline{21}} + e^{\overline{12}} \rangle$$

we see that $\ker \mathcal{P}' \otimes \mathcal{Q}$ is composed of vectors which are annihilated by the following three functionals:

$$\{e^{11}, e^{22}, e^{21} + qe^{12}\} \otimes (\overline{q}e^{\overline{12}} - e^{\overline{21}})$$

$$= \{\overline{q}e^{11\overline{12}} - e^{11\overline{21}}, \overline{q}e^{22\overline{12}} - e^{22\overline{21}}, |q|^2e^{12\overline{12}} + \overline{q}e^{21\overline{12}} - qe^{12\overline{21}} - e^{21\overline{21}}\}$$

(here $e^{11\overline{12}} := e^{11} \otimes e^{\overline{12}}$, etc) and $\ker \mathcal{P} \otimes \mathcal{Q}'$ is composed of vectors which are annihilated by the following three functionals:

$$(qe^{21} - e^{12}) \otimes \{e^{\overline{11}}, e^{\overline{22}}, \overline{q}e^{\overline{21}} + e^{\overline{12}}\}$$

$$= \{qe^{21\overline{11}} - e^{12\overline{11}}, qe^{21\overline{22}} - e^{12\overline{22}}, |q|^2e^{21\overline{21}} + qe^{21\overline{12}} - \overline{q}e^{12\overline{21}} - e^{12\overline{12}}\}.$$

Composing all the six functionals with X_{23}^{-1}, we obtain the following functionals:

$$\{\overline{q}(e^{1\bar{1}1\bar{2}} - \varepsilon e^{1\bar{2}2\bar{2}}) - te^{1\bar{2}1\bar{1}}, \overline{q}te^{2\bar{1}2\bar{2}} - e^{2\bar{2}2\bar{1}}, |q|^2te^{1\bar{1}2\bar{2}} + \overline{q}(e^{2\bar{1}1\bar{2}} - \varepsilon e^{2\bar{2}2\bar{2}}) - qe^{1\bar{2}2\bar{1}} - te^{2\bar{2}1\bar{1}},$$

$$q(e^{2\bar{1}1\bar{1}} - \varepsilon e^{2\bar{2}2\bar{1}}) - te^{1\bar{1}2\bar{1}}, qte^{2\bar{2}1\bar{2}} - e^{1\bar{2}2\bar{2}}, |q|^2te^{2\bar{2}1\bar{1}} + q(e^{2\bar{1}1\bar{2}} - \varepsilon e^{2\bar{2}2\bar{2}})$$

$$-\overline{q}e^{1\bar{2}2\bar{1}} - te^{1\bar{1}2\bar{2}}\}$$

and equation (11) is equivalent to the vanishing of these functionals on $x_{12}x_{34}$. This way we obtain the following six relations:

$$\overline{q}(x^{1\bar{1}}x^{1\bar{2}} - \varepsilon x^{1\bar{2}}x^{2\bar{2}}) - tx^{1\bar{2}}x^{1\bar{1}} = 0$$

$$\overline{q}tx^{2\bar{1}}x^{2\bar{2}} - x^{2\bar{2}}x^{2\bar{1}} = 0$$

$$|q|^2tx^{1\bar{1}}x^{2\bar{2}} + \overline{q}(x^{2\bar{1}}x^{1\bar{2}} - \varepsilon x^{2\bar{2}}x^{2\bar{2}}) - qx^{1\bar{2}}x^{2\bar{1}} - tx^{2\bar{2}}x^{1\bar{1}} = 0$$

$$q(x^{2\bar{1}}x^{1\bar{1}} - \varepsilon x^{2\bar{2}}x^{2\bar{1}}) - tx^{1\bar{1}}x^{2\bar{1}} = 0$$

$$qtx^{2\bar{2}}x^{1\bar{2}} - x^{1\bar{2}}x^{2\bar{2}} = 0$$

$$|q|^2tx^{2\bar{2}}x^{1\bar{1}} + q(x^{2\bar{1}}x^{1\bar{2}} - \varepsilon x^{2\bar{2}}x^{2\bar{2}}) - \overline{q}x^{1\bar{2}}x^{2\bar{1}} - tx^{1\bar{1}}x^{2\bar{2}} = 0.$$

Substituting

$$\begin{pmatrix} x^{1\bar{1}} & x^{1\bar{2}} \\ x^{2\bar{1}} & x^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{A7}$$

we can write these relations as follows:

$$\overline{q}(\alpha\beta - \varepsilon\beta\delta) - t\beta\alpha = 0 \tag{A8}$$

$$\overline{q}t\gamma\delta - \delta\gamma = 0 \tag{A9}$$

$$|q|^2t\alpha\delta + \overline{q}(\gamma\beta - \varepsilon\delta\delta) - q\beta\gamma - t\delta\alpha = 0 \tag{A10}$$

$$q(\gamma\alpha - \varepsilon\delta\gamma) - t\alpha\gamma = 0 \tag{A11}$$

$$qt\delta\beta - \beta\delta = 0 \tag{A12}$$

$$|q|^2t\delta\alpha + q(\gamma\beta - \varepsilon\delta\delta) - \overline{q}\beta\gamma - t\alpha\delta = 0. \tag{A13}$$

Now we shall show that for the PBW theorem, condition (12) is necessary. We thus consider case 1, i.e. $\varepsilon = 0$. In this case, the commutation relations take the form

$$\beta\alpha = \overline{q}t^{-1}\alpha\beta \tag{A14}$$

$$\gamma\alpha = q^{-1}t\alpha\gamma \tag{A15}$$

$$\delta\gamma = \overline{q}t\gamma\delta \tag{A16}$$

$$\delta\beta = q^{-1}t^{-1}\beta\delta \tag{A17}$$

$$|q|^2t\delta\alpha + q\gamma\beta = t\alpha\delta + \overline{q}\beta\gamma \tag{A18}$$

$$-t\delta\alpha + \overline{q}\gamma\beta = -|q|^2t\alpha\delta + q\beta\gamma. \tag{A19}$$

Taking $\overline{q}$(A18) $- q$(A19) and (A18) $+ |q|^2$(A19) instead of (A18) and (A19), we obtain

$$q(\overline{q}^2+1)t\delta\alpha = \overline{q}(q^2+1)t\alpha\delta + (\overline{q}^2 - q^2)\beta\gamma \tag{A20}$$

$$q(\overline{q}^2+1)\gamma\beta = t(1-|q|^4)\alpha\delta + \overline{q}(q^2+1)\beta\gamma. \tag{A21}$$

Using equations (A14)–(A17) and (A20), (A21) it is easy to see that each element of the algebra can be written as a sum of (alphabetically) ordered monomials in $\alpha, \beta, \gamma, \delta$. Now, if we perform the two independent ways of ordering of $q(\overline{q}^2+1)\gamma\beta\alpha$, we obtain

$$\begin{aligned} q(\overline{q}^2+1)\gamma(\beta\alpha) &= q(\overline{q}^2+1)\overline{q}t^{-1}\gamma\alpha\beta = \overline{q}(\overline{q}^2+1)\alpha\gamma\beta \\ &= \frac{\overline{q}}{q}\alpha[t(1-|q|^4)\alpha\delta + \overline{q}(q^2+1)\beta\gamma] \end{aligned}$$

on the one hand, and

$$\begin{aligned} q(\overline{q}^2+1)(\gamma\beta)\alpha &= [t(1-|q|^4)\alpha\delta + \overline{q}(q^2+1)\beta\gamma]\alpha \\ &= \frac{1-|q|^4}{q(\overline{q}^2+1)}\alpha[t\overline{q}(q^2+1)\alpha\delta + (\overline{q}^2-q^2)\beta\gamma] + \frac{\overline{q}}{q}t(q^2+1)\beta\alpha\gamma \\ &= (1-|q|^4)\frac{\overline{q}}{q}\frac{q^2+1}{\overline{q}^2+1}t\alpha\alpha\delta + \frac{(1-|q|^4)(\overline{q}^2-q^2)}{q(\overline{q}^2+1)}\alpha\beta\gamma + \frac{\overline{q}^2}{q}(q^2+1)\alpha\beta\gamma \end{aligned}$$

on the other. Comparing the coefficients at $\alpha\alpha\delta$ we obtain (12). Comparing at $\alpha\beta\gamma$ gives exactly the same. (We assume, of course, that $\alpha\alpha\delta$ and $\alpha\beta\gamma$ are linearly independent.)

For $|q| = 1$, relations (A20) and (A21) are equivalent to

$$\gamma\beta = \beta\gamma \qquad [\alpha, \delta] = \frac{1}{t}(q - q^{-1})\beta\gamma \tag{A22}$$

and the algebra Poly (V) resembles the usual $GL_q(2)$ algebra (it is the same, if $t = 1$). One can easily check that Poly (V) is a q-enveloping algebra in the sense of [13], if we order the generators as follows:

$$e_1 := \alpha \qquad e_2 := \beta \qquad e_3 := \gamma \qquad e_4 := \delta$$

hence the PBW theorem holds in this case (see [13, theorem 2.8.1]).

If $\overline{q} = q$, relations (A20) and (A21) are equivalent to

$$\delta\alpha = \alpha\delta \qquad [\beta, \gamma] = t(q - q^{-1})\alpha\delta.$$

Replacing $\alpha \leftrightarrow \beta$, $\gamma \leftrightarrow \delta$, $t \leftrightarrow t^{-1}$, we obtain the same relations as in the previous case, hence the PBW theorem holds also in this case. Similarly, it holds for $\overline{q} = -q$.

The case when $\varepsilon = \pm 1$ and $q = t > 0$ corresponds to the standard quantum deformation of the Lorentz group, containing as a subgroup $SU_q(2)$ or $SU_q(1,1)$ (depending on the sign of $\varepsilon(q-1)$). Relations (A8)–(A13) are then equivalent to those considered by many authors [14–16]. It can be easily shown that the PBW theorem holds in this case, using the Diamond lemma [17] (choose $\beta < \alpha < \delta < \gamma$ as the total ordering).

A.3. The algebra $\mathcal{B}$

We set $\mathcal{A} := \mathrm{Poly}\,(H)$, $\mathcal{C} := \mathrm{Poly}\,(V)$

The uniqueness of s is obvious, since its value on any monomial can be reduced by (26) to the case (19).

Writing equation (19) as

$$x_{12}u_3 = T u_1 x_{23}$$

(in case the crossed product exists), we obtain

$$x_{12}E_{34} = x_{12}u_3u_4E_{34} = T_{123}u_1x_{23}u_4E_{14} = T_{123}T_{234}u_1u_2x_{34}E_{12} = T_{123}T_{234}E_{12}x_{34} \tag{A23}$$

hence T must satisfy

$$T_{123}T_{234}E_{12} = E_{34}. \tag{A24}$$

Taking into account that $X_{23}X_{12}E_{23} = E_{12}$, this means that

$$S_{12}S_{23}E_{12} = E_{23}. \tag{A25}$$

It is easy to see that the only solutions of (A25) which are intertwiners of u_1u_2 (hence of the form $aI + bEE'$) are (29).

Conversely, we shall show that if S is given by (29) then there exists $s : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ with the required properties. Let $\widetilde{\mathcal{A}}$ ($\widetilde{\mathcal{C}}$) be the free *-algebra generated by u^A_B ($x^{A\overline{B}}$). We have

$$\mathcal{A} = \widetilde{\mathcal{A}}/\mathcal{J}_{\mathcal{A}} \qquad \mathcal{C} = \widetilde{\mathcal{C}}/\mathcal{J}_{\mathcal{C}} \tag{A26}$$

where $\mathcal{J}_{\mathcal{A}} = \langle \mathcal{J}^0_{\mathcal{A}} \rangle$ is the ideal generated by $\mathcal{J}^0_{\mathcal{A}} := \{u_1u_2E - E,\, E'u_1u_2 - E',\, Xu_1\overline{u}_2 - \overline{u}_1u_2X\}$ in $\widetilde{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{C}} = \langle \mathcal{J}^0_{\mathcal{C}} \rangle$ is the ideal generated by $\mathcal{J}^0_{\mathcal{C}} := \{\hat{R}x_{\mathbf{1}}x_{\mathbf{2}} - x_{\mathbf{1}}x_{\mathbf{2}},\, (x^{A\overline{B}})^* = x^{B\overline{A}}\}$ in $\widetilde{\mathcal{C}}$. Here $\hat{R} = \hat{R}_-$ for $\overline{q} = q$ and $\hat{R} = \hat{R}_+$ for $|q| = 1$ (cf the discussion near (17)). It is easy to see that there exists a (unique) map $\widetilde{s} : \widetilde{\mathcal{C}} \otimes \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{C}}$ satisfying (19) and (26), with $\mathcal{A}, \mathcal{C}, s$ replaced by $\widetilde{\mathcal{A}}, \widetilde{\mathcal{C}}, \widetilde{s}$.

The proof will be complete if we show that

$$\widetilde{s}(\widetilde{\mathcal{C}} \otimes \mathcal{J}_{\mathcal{A}}) \subset \mathcal{J}_{\mathcal{A}} \otimes \widetilde{\mathcal{C}} \qquad \widetilde{s}(\mathcal{J}_{\mathcal{C}} \otimes \widetilde{\mathcal{A}}) \subset \widetilde{\mathcal{A}} \otimes \mathcal{J}_{\mathcal{C}}.$$

Since $\{a \in \widetilde{\mathcal{A}} : \widetilde{s}(\widetilde{\mathcal{C}} \otimes a) \subset \mathcal{J}_{\mathcal{A}} \otimes \widetilde{\mathcal{C}}\}$ is an ideal in $\widetilde{\mathcal{A}}$, it is sufficient to show that

$$\widetilde{s}(\widetilde{\mathcal{C}} \otimes \mathcal{J}^0_{\mathcal{A}}) \subset \mathcal{J}_{\mathcal{A}} \otimes \widetilde{\mathcal{C}} \tag{A27}$$

and, similarly,

$$\widetilde{s}(\mathcal{J}^0_{\mathcal{C}} \otimes \widetilde{\mathcal{A}}) \subset \widetilde{\mathcal{A}} \otimes \mathcal{J}_{\mathcal{C}}. \tag{A28}$$

We shall show that

$$\widetilde{s}(\widetilde{\mathcal{C}}^{(1)} \otimes \mathcal{J}^0_{\mathcal{A}}) \subset \mathcal{J}_{\mathcal{A}} \otimes \widetilde{\mathcal{C}}^{(1)} \qquad \widetilde{s}(\mathcal{J}^0_{\mathcal{C}} \otimes \widetilde{\mathcal{A}}^{(1)}) \subset \widetilde{\mathcal{A}}^{(1)} \otimes \mathcal{J}_{\mathcal{C}} \tag{A29}$$

where $\widetilde{\mathcal{A}}^{(1)}$ and $\widetilde{\mathcal{C}}^{(1)}$ denote the linear subspaces spanned by the corresponding generators. This is sufficient, because then from (26) it follows that

$$\widetilde{s}(\widetilde{\mathcal{C}}^{(n)} \otimes \mathcal{J}^0_{\mathcal{A}}) \subset \mathcal{J}_{\mathcal{A}} \otimes \widetilde{\mathcal{C}}^{(n)} \qquad \widetilde{s}(\mathcal{J}^0_{\mathcal{C}} \otimes \widetilde{\mathcal{A}}^{(n)}) \subset \widetilde{\mathcal{A}}^{(n)} \otimes \mathcal{J}_{\mathcal{C}} \tag{A30}$$

where $\widetilde{\mathcal{A}}^{(n)}$ and $\widetilde{\mathcal{C}}^{(n)}$ denote the subspaces spanned by monomials of order n.

To show (A29), note that

$$\begin{aligned}x_{12}(u_3u_4E_{34}-E_{34}) &= T_{123}T_{234}u_1u_2x_{34}E_{12}-x_{12}E_{34}\\ &= T_{123}T_{234}(u_1u_2E_{12}-E_{12})x_{34}+T_{123}T_{234}E_{12}x_{34}-x_{12}E_{34}\\ &= T_{123}T_{234}(u_1u_2E_{12}-E_{12})x_{34}\end{aligned}$$

belongs to $\mathcal{J}^0_{\mathcal{A}}\otimes\widetilde{\mathcal{C}}$. Similarly, $x_{12}(E'_{34}u_3u_4-E'_{34})\in\mathcal{J}^0_{\mathcal{A}}\otimes\widetilde{\mathcal{C}}$ and

$$x_{12}(X_{34}u_3\overline{u}_4-\overline{u}_3u_4)=X_{34}T_{123}T'_{234}u_1\overline{u}_2x_{34}-T'_{123}T_{234}\overline{u}_1u_2x_{34}X_{12}=0$$

where $T'=(\tau\overline{S}^{-1}\tau)_{23}X^{-1}_{12}$ is the matrix appearing in (23). The equality $X_{34}T_{123}T'_{234}=T'_{123}T_{234}X_{12}$ is proved using formulae of the type (A1)–(A6). Furthermore, we have

$$\begin{aligned}(\hat{R}_{1234}x_{12}x_{34}-x_{12}x_{34})u_5 &= \hat{R}_{1234}x_{12}T_{345}u_3x_{45}-x_{12}T_{345}u_3x_{45}\\ &= \hat{R}_{1234}T_{345}T_{123}u_1x_{23}x_{45}-T_{345}T_{123}u_1x_{23}x_{45}\\ &= T_{345}T_{123}u_1(\hat{R}_{2345}x_{23}x_{45}-x_{23}x_{45})\in\widetilde{\mathcal{A}}\otimes\mathcal{J}^0_{\mathcal{C}}\end{aligned}$$

since $\hat{R}_{1234}T_{345}T_{123}=T_{345}T_{123}\hat{R}_{2345}$ (this also follows from (A1)–(A6)).

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